INFORMATION AND CONTROL 19, 319–344 (1971)

# Reachable Sets for Linear Dynamical Systems

THOMAS PECSVARADI

*NASA, Ames Research Center, Moffett Field, California 94035*

AND

KUMPATI S. NARENDRA

*Becton Laboratory, Yale University, New Haven, Connecticut 06520*

The properties of reachable sets for linear dynamical systems for specified control sets are discussed. Iterative procedures for determining numerical approximations of the reachable set are suggested and methods of obtaining an admissible control function which transfers an initial state to as near a prescribed target as possible is described. The problem of reachability with multiple control constraints is discussed and certain aspects of reachability for time-invariant systems with adjustable parameters is considered.

## 1. INTRODUCTION

If the admissible controls of a linear dynamical system are constrained in some sense, then the transfer of arbitrary initial states to arbitrary terminal states is generally not possible. Given an initial state, the set of all terminal states to which the system can be transferred is referred to as the reachable set under the specified control constraint. The problem of reachability was originally suggested by Roxin (1960) who investigated reachable sets for nonlinear autonomous systems in which the control function appears linearly. LeMay (1964) derived theorems relating the reachable and controllable sets for linear time-varying systems with bounded inputs and also derived necessary and sufficient conditions for the controllable set to be the entire state space. A procedure was presented by Formalskii (1967) for constructing the controllable set for a linear time-invariant system with a bounded impulse scalar control. Neustadt (1963) studied the reachable sets for time-varying systems which are linear with respect to the state but nonlinear with respect to the control and considered admissible control functions which are restricted to lie in a compact subspace of the $m$-dimensional Euclidean space.

In this paper the set of admissible controls is considered to be a subset of the $L_p(t_0, t_1)$ space. Specifically, the admissible controls are Lebesgue measurable functions whose $L_p(1 \leqslant p \leqslant \infty)$ norms are bounded. The purpose of the paper is

(i) to investigate some of the relevant properties of reachable sets,

(ii) to develop iterative procedures for the solution of various classes of control problems.

The paper is organized as follows. The control sets and the corresponding reachable sets are defined and their properties are briefly discussed in Section 2. Section 3 describes an iterative procedure for finding an admissible control function which transfers an initial state to as near a prescribed target as possible. In Section 4 an iterative scheme is presented to obtain a numerical approximation of the reachable set. The problem of reachability with multiple control constraints is discussed in Section 5 and an algorithm is described for the solution of a class of optimal control problems. In Section 6, certain aspects of reachability for time-invariant systems with adjustable parameters are considered, and finally in Section 7 several illustrative examples are given.

The following notation will be used throughout the paper: $E^n$ denotes the Euclidean $n$-space; if $x, y \in E^n$, then their inner product is denoted by $\langle x, y \rangle = x^T y$, where $x^T$ denotes the transpose of $x$; for any $x \in E^n$, the norm is denoted by $\| x \|$ and $\| x \| = \sqrt{\langle x, x \rangle}$; if $y_1, y_2, \ldots, y_p \in E^n$, then their closed convex hull is denoted by $\Delta(y_1, y_2, \ldots, y_p)$. A set $S$ is said to be symmetric about the point $x_0 \in S$ if $(x_0 + x) \in S$ implies $(x_0 - x) \in S$. The boundary and the interior of the set $S$ are denoted by $\partial S$ and int $S$, respectively.

## 2. Reachable Sets

*The System*

Consider a linear dynamical system described by the vector differential equation

$$\dot{x} = A(t)x + B(t)u, \tag{1}$$

where $x$ is an $n$-vector representing the state of the system, $u$ is an $m$-vector representing the control or input to the system, and $A(t)$ and $B(t)$ are $(n \times n)$ and $(n \times m)$ time-varying matrices, respectively. The space of inputs is the function space $L_p(t_0, t_1)$ for various values of $p$, $1 \leqslant p \leqslant \infty$, and arbitrary finite intervals of time $(t_0, t_1)$. Thus, for a given problem every admissible control $u(t)$ must satisfy the relation

$$u \in U \subset L_p(t_0, t_1), \tag{2}$$

for some value of $p$ and some set $U$.

*The Control Sets U.* Throughout this paper, the admissible controls $u(t)$ for a given problem are considered to be elements of a representative constraint set $U_p \subset L_p(t_0, t_1)$ defined by

$$U_p = \{u \in L_p(t_0, t_1) : \| u \|_p \leqslant c_p, 0 < c_p < \infty\}, \tag{3}$$

where $1 \leqslant p \leqslant \infty$, $c_p$ is an arbitrary positive number, and $\| u \|_p$ represents the $L_p$-norm of the $m$-vector function $u(t)$, viz.,

$$\| u \|_p = \left[ \int_{t_0}^{t_1} \sum_{j=1}^{m} | u_j(t) |^p \, dt \right]^{1/p}. \tag{4}$$

The cases when $p = 1, 2$, and $\infty$ correspond to fuel, energy, and amplitude constraints, respectively.

In some problems the admissible controls may be required to satisfy multiple constraints of the form (4) simultaneously (Section 5). For example, assume that for a given problem the admissible controls are constrained to lie in $U_q$, $U_r$, and $U_s$, $q < r < s$, simultaneously. Since $L_{p+1}(t_0, t_1) \subset L_p(t_0, t_1)$ for all $p$, $1 \leqslant p < \infty$, the effective control set $U$ is given by

$$U = U_q \cap U_r \cap U_s = \{u \in L_s(t_0, t_1) : \| u \|_q \leqslant c_q, \| u \|_r \leqslant c_r, \| u \|_s \leqslant c_s\}. \tag{5}$$

ASSUMPTIONS. The following assumptions are made throughout the paper:

(i) Every element of the matrix $A(t)$ is integrable on each finite time interval $(t_0, t_1)$.

(ii) If the admissible controls for a given problem are elements of the $L_p(t_0, t_1)$ space, then every element of the matrix $B(t)$ is an element of the $L_{p'}(t_0, t_1)$ space, where

$$p' \geqslant p/(p - 1).$$

(iii) $L_1(t_0, t_1)$ denotes that space of measures which make up the bounded linear functionals on $L_\infty(t_0, t_1)$.

(iv) If the admissible controls for a given problem are elements of the $L_1(t_0, t_1)$ space, then every element of the matrix $B(t)$ is continuous.

By Assumption (i) the transition matrix of system (1) is unique, nonsingular, and absolutely continuous. In light of this, Assumption (ii) guarantees the existence of a solution of Eq. (1). According to Assumption (iii), the space $L_1(t_0, t_1)$ contains not only the Lebesgue integrable functions on $(t_0, t_1)$, but also the symbolic or generalized function $\delta(t - \tau)$, $t_0 < \tau < t_1$, commonly referred to as the "delta-function". Assumptions (iii) and (iv) assure the existence of a solution to the optimization problems of Sections 4 and 5.

Definition 1 (reachable state). Consider the system (1) with initial state $x(t_0) = x_0$ and admissible controls which satisfy relation (2) on every finite time interval $(t_0, t_1)$ for a given control set $U$. The state $x_1 \in E^n$ is said to be reachable at time $t_1$ from $x(t_0) = x_0$ if there exists an admissible control and a time $t_1 \geqslant t_0$ such that the corresponding solution of Eq. (1) at time $t_1$ coincides with $x_1$, i.e.,

$$x(t_1) = \Phi(t_1, t_0)\, x_0 + \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, u(t)\, dt = x_1 \quad \text{for some } u \in U. \tag{6}$$

Definition 2 (reachable set). The reachable set $R(t_1; x_0) \subset E^n$ at time $t_1$ for the system (1) with control constraint set $U$ is defined as the set of all states $x \in E^n$ reachable at time $t_1$ from $x(t_0) = x_0$ by admissible controls, viz.,

$$R(t_1; x_0) = \left\{ x \in E^n : x = \Phi(t_1, t_0)\, x_0 + \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, u(t)\, dt,\ u \in U \right\}. \tag{7}$$

The set $R(t_1; x_0)$ is a rigid translate of the set $R(t_1; 0)$, which is denoted for convenience by $R(t_1)$, i.e.,

$$R(t_1) = \left\{ x \in E^n : x = \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, u(t)\, dt,\ u \in U \right\}. \tag{8}$$

*Properties of the Reachable Sets*

Since the control sets treated in this paper are symmetric about the null function $u(t) \equiv 0$ on $(t_0, t_1)$, the reachable sets $R(t_1)$ are symmetric about the origin of $E^n$.

If system (1) is time-invariant, then the set $R(t_1)$ grows monotonically with $t_1$, i.e.,

$$R(t_1) \subset R(t_2) \qquad \text{for} \quad t_1 \leqslant t_2. \tag{9}$$

In order to show that relation (9) holds, assume that $x_1 \in R(t_1)$ is an arbitrary point in $R(t_1)$. Then there exists an admissible control $u_1 \in U$ such that

$$x_1 = \int_{t_0}^{t_1} \Phi(t_1 - t)\, Bu_1(t)\, dt.$$

Let $t_2 = t_1 + \sigma$, $\sigma > 0$, and define the control $u_2(t)$ by

$$u_2(t) = \begin{cases} 0 & t_0 \leqslant t < t_0 + \sigma, \\ u_1(t - \sigma) & t_0 + \sigma \leqslant t \leqslant t_2 . \end{cases}$$

Clearly, $u_2(t)$ is admissible on $(t_0, t_2)$. The corresponding point $x_2 \in R(t_2)$ is

$$x_2 = \int_{t_0}^{t_2} \Phi(t_2 - t)\, Bu_2(t)\, dt = \int_{t_0+\sigma}^{t_2} \Phi(t_2 - t)\, Bu_1(t - \sigma)\, dt.$$

Introducing the change of variable $\tau = t - \sigma$, this becomes

$$x_2 = \int_{t_0}^{t_1} \Phi(t_1 - \tau)\, Bu_1(\tau)\, d\tau = x_1 .$$

Thus, if $x_1 \in R(t_1)$, then $x_1 \in R(t_2)$, $t_1 \leqslant t_2$. Since this is true for any $x_1 \in R(t_1)$, relation (9) is proved. If the system is time-varying, then relation (9) does not necessarily hold.

*Method of Support Hyperplane*

In general, the set $R(t_1)$ cannot be characterized explicitly in terms of inequalities of the form

$$\Psi_i(x) \leqslant 0, \qquad i = 1, 2, \ldots, N,$$

but must be computed from its implicit definition (8). It seems reasonable therefore to try to characterize $R(t_1)$ by its boundary points. It will now be shown that the computation of a boundary point of $R(t_1)$ is equivalent to the solution of a straightforward optimal control problem.

Let $h$ be an arbitrary nonzero vector in $E^n$. Then the support function $\eta(h)$ of $R(t_1)$ is defined as

$$\eta(h) = \max_{x \in R(t_1)} \langle h, x \rangle \tag{10}$$

and the support hyperplane $P(h)$ of $R(t_1)$ with outward normal $h$ is given by

$$P(h) = \{x \in E^n : \langle h, x \rangle = \eta(h)\}. \tag{11}$$

A contact point $s(h) \in \partial R(t_1)$ is defined by the relation

$$\langle h, s(h) \rangle = \eta(h). \tag{12}$$

Clearly, $s(h) \in P(h) \cap R(t_1)$ and is unique if and only if $R(t_1)$ is strictly convex.

In order to obtain a boundary point of $R(t_1)$ it is necessary to solve the maximization problem indicated in (10). This is done by converting (10) into the following optimal control problem: Given the system (1) with initial state $x(t_0) = 0$ and a control set $U_p$ of the form (3), find an admissible control function $u(t, h)$ which maximizes the performance index

$$J = \langle h, x(t_1) \rangle = \int_{t_0}^{t_1} f^T(t, h)\, u(t)\, dt, \tag{13}$$

where $f(t, h)$ is the $m$-vector-valued function defined as

$$f(t, h) = B^T(t)\, \Phi^T(t_1, t) h, \tag{14}$$

and $h$ is a fixed nonzero vector in $E^n$. It can be easily shown that $u(t, h)$ lies on the boundary of $U_p$, i.e. $u(h) \in U_p{}^0$, where $U_p{}^0$ is defined by

$$U_p{}^0 = \{u \in L_p(t_0, t_1) : \| u \|_p = c_p\}. \tag{15}$$

Applying the Maximum Principle when $1 < p \leqslant \infty$ and standard techniques of functional analysis when $p = 1$, it can be shown [9] that $u(t, h)$ is of the form

$1 < p < \infty$:

$$u_j(t, h) = \frac{c_p}{\| f(h) \|_q^{1/(p-1)}} \, | f_j(t, h) |^{1/(p-1)} \operatorname{sgn}(f_j(t, h)), \qquad j = 1, 2, \ldots, m. \tag{16}$$

$p = 1$:

$$u_j(t, h) = \begin{cases} 0 & j = 1, 2, \ldots, m; \quad j \neq k, \\ c_p \operatorname{sgn}(f_k(\tau_k, h))\, \delta(t - \tau_k), & j = k, \end{cases}$$

$$\text{where} \quad | f_k(\tau_k, h) | = \max_{\substack{1 \leqslant j \leqslant m \\ t_0 \leqslant t \leqslant t_1}} | f_j(t, h) |. \tag{17}$$

$p = \infty$:

$$u_j(t, h) = c_p \operatorname{sgn}(f_j(t, h)), \qquad j = 1, 2, \ldots, m. \tag{18}$$

For $p = 1$ $u(t, h)$ may not be unique since $\max | f_j(t, h) |$ may occur for several values of $t$ and $j$.

THEOREM 1. *Consider system* (1) *with initial state* $x(t_0) = 0$ *and control set* $U = U_p$, $1 \leqslant p \leqslant \infty$, *and let the Assumptions* (i)–(iv) *be satisfied. Then the reachable set* $R(t_1)$ *given by* (8) *is convex, compact, and grows strictly monotonically with* $c_p$ *for all* $p$ *in* $1 \leqslant p \leqslant \infty$.

*Proof.* Convexity follows directly from the fact that $U_p$ is convex and $R(t_1)$ is a linear mapping of $U_p$. Compactness is proved by showing that $R(t_1)$ is a closed and bounded set in $E^n$. Combining the expressions for $u(t, h)$ with Eqs. (8) and (13) yields

$$\eta(h) = c_p \|f(h)\|_q, \qquad q = p/(p-1), \tag{19}$$

where if $p = 1$, then $q = \infty$; if $p = \infty$, then $q = 1$. Assumptions (i)–(iv) and the fact $L_{p+1}(t_0, t_1) \subset L_p(t_0, t_1)$ for all $p$, $1 \leqslant p < \infty$ imply that $f(h) \in L_q(t_0, t_1)$. Consequently, $\eta(h) < \infty$, and therefore $R(t_1)$ is bounded for all $p$, $1 \leqslant p \leqslant \infty$.

Since $u(t, h)$ given by Eq. (16)–(18) are admissible, the corresponding contact points $s(h)$ are elements of $R(t_1)$. But $s(h) \in \partial R(t_1)$. Hence, $R(t_1)$ contains all of its boundary points and therefore is closed.

The strictly monotonic growth of $R(t_1)$ with $c_p$ is obvious from the strictly monotonic growth of $\eta(h)$ with $c_p$ as shown by Eq. (19).

THEOREM 2. *Consider system* (1) *with the matrices* $A$ *and* $B$ *constant, and let* $\lambda_i$, $i = 1, 2,..., n$ *denote the eigenvalues of* $A$. *Let the initial state be* $x(t_0) = 0$, *the control set* $U = U_p$, $1 \leqslant p \leqslant \infty$, *and let the Assumptions* (iii) *and* (iv) *be satisfied* (*Assumptions* (i) *and* (ii) *are automatically satisfied*). *The infinite-time reachable set* $R(\infty)$ *is defined as*

$$R(\infty) = \lim_{t_1 \to \infty} R(t_1). \tag{20}$$

*If* $\mathrm{Re}(\lambda_i) < 0$ *for all* $i = 1, 2,..., n$, *then* $R(\infty)$ *is bounded for all* $p$ *in* $1 \leqslant p \leqslant \infty$.

*Proof.* The proof is established by showing that

$$\lim_{t_1 \to \infty} \eta(h) < \infty \qquad \text{for all} \quad h \in E^n \tag{21}$$

Consider the case when $1 < p \leqslant \infty$. The expression for $\eta(h)$ is

$$\eta(h) = c_p \|f(h)\|_q = c_p \left[\int_{t_0}^{t_1} \sum_{j=1}^{m} |f_j(t, h)|^q \, dt\right]^{1/q}, \qquad q = p/(p-1),$$

where for the autonomous case under consideration

$$f(t, h) = B^T e^{A^T(t_1 - t)} h.$$

Since $\mathrm{Re}(\lambda_i) < 0$ for all $i = 1, 2, \ldots, n$, the following bound exists for $f(t, h)$

$$\sum_{j=1}^{m} |f_j(t, h)|^q < C_1^q e^{-\xi(t_1 - t)} \qquad \text{for all} \quad t \in (0, t_1),$$

where $\xi$ and $C_1$ are positive constants. Thus,

$$\eta(h) < c_p C_1 \left[ \int_{t_0}^{t_1} e^{-\xi(t_1 - t)} \, dt \right]^{1/q}$$

$$= c_p C_1 \left[ \frac{1 - e^{-\xi t_1}}{\xi} \right]^{1/q}.$$

Passing to the limit as $t_1 \to \infty$ yields

$$\lim_{t_1 \to \infty} \eta(h) = \frac{c_p C_1}{\xi^{1/q}} < \infty.$$

When $p = 1$, $\eta(h)$ is given by

$$\eta(h) = c_p \, |f_k(\tau_k, h)| \leqslant c_p C_1 e^{-\xi(t_1 - \tau_k)} \leqslant c_p C_1 < \infty.$$

## 3. Reachability of a Given State

The aim of a very general optimal control problem is to find an admissible control which transfers the initial state of a dynamical system to a prescribed target while minimizing a given performance index. When a designer is faced with such a problem, he generally has no information concerning the existence of an admissible control that does accomplish the desired transfer. Various methods can be applied to determine whether or not the target is reachable by an admissible control. All these methods are iterative in nature and involve the minimization of the Euclidean distance from the target to the reachable set.

The original method was suggested by Gilbert (1966) who generated a sequence of line segments, each member of which lies completely in the reachable set, and then minimized the distance between the target and these line segments. Barr (1967, 1969) extended Gilbert's procedure so that the minimization is carried out over a sequence of convex polyhedra. The authors in 1969 developed a new procedure as an alternative to that suggested by Barr. It is this latter procedure which will now be used to determine whether or not a prescribed target is reachable. Consider the following problem.

*Problem.* Given the system (1) with initial state $x(t_0) = 0$, terminal target $x(t_1) = \alpha$, and control set $U = U_p$, $1 \leqslant p \leqslant \infty$, find an admissible control $u^*(t)$ which transfers the state of system (1) from the origin to a point $x^* \in R(t_1)$ such that

$$\| \alpha - x^* \| = \min_{x \in R(t_1)} \| \alpha - x \|. \tag{22}$$

The following iterative procedure generates a sequence of control functions $\{u(t, h_k)\}$ and a sequence of points $\{x_k\}$ such that $u(t, h_k) \to u^*(t)$ and $x_k \to x^*$.

*The Iterative Procedure*

(i) $1 \leqslant k < n$: Let $u_0(t), u_1(t),..., u_{k-1}(t)$ be $k$ known admissible controls and define the points $y_0, y_1,..., y_{k-1} \in R(t_1)$ as

$$y_i = \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, u_i(t)\, dt, \qquad i = 0, 1,..., k-1. \tag{23}$$

Let $Q_k = \Delta(y_0, y_1,..., y_{k-1})$ and compute the point $x_{k-1} \in Q_k$ such that

$$\| \alpha - x_{k-1} \| = \min_{x \in Q_k} \| \alpha - x \|. \tag{24}$$

Let $h_k = \alpha - x_{k-1}$ and use Eq. (16), (17), or (18) to find the control $u(t, h_k)$. Then the control $u_k(t)$ is defined as

$$u_k(t) = u(t, h_k). \tag{25}$$

(ii) $k \geqslant n$: Let $u_0(t), u_1(t),..., u_{n-1}(t)$ be $n$ known admissible controls, define the points $y_i \in R(t_1)$, $i = 0, 1,..., n-1$, by Eq. (23) and let $Q_k = \Delta(y_0, y_1,..., y_{n-1})$. Compute $x_{k-1} \in Q_k$, $h_k$, and $u(t, h_k)$ as in part (i) and define $u_n(t)$ and $y_n$ as

$$u_n(t) = u(t, h_k) \qquad \text{and} \qquad y_n = \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, u_n(t)\, dt = s(h_k). \tag{26}$$

The convex hull $S_k$ and the point $x_k \in S_k$ are then defined as

$$S_k = \Delta(Q_k, y_n), \tag{27}$$

$$\| \alpha - x_k \| = \min_{x \in S_k} \| \alpha - x \|. \tag{28}$$

Let $P_{k+1}$ denote the hyperplane containing $x_k$ whose normal is $h_{k-1}$, and let $d_i$, $i = 0, 1,..., n-1$, be the Euclidean distance from $y_i$ to $P_{k+1}$. Assume that $\max_{0 \leqslant i \leqslant n} d_i$ occurs for $i = m$. Then the control $u_m(t)$ is replaced by $u_n(t)$. In order to start the iterative procedure, choose $u_0(t) \equiv 0$ on $(t_0, t_1)$.

The results of the Theorem in [Pecsvaradi and Narendra, 1970] imply that $x_k \to x^*$ as $k \to \infty$. If at some stage, say $\bar{k}$, the iterative procedure terminates, i.e., $x_{\bar{k}} = x_{\bar{k}-1} = x^*$, then $x^*$ can be expressed as the convex combination of the $y_i$, $i = 0, 1, \ldots, n$, viz.,

$$x^* = y_0 + \sum_{i=1}^{n} \mu_i(y_i - y_0), \tag{29}$$

where the $\mu_i$ are constants satisfying the conditions

$$\sum_{i=1}^{n} \mu_i \leqslant 1 \qquad \text{and} \qquad 0 \leqslant \mu_i \leqslant 1, \qquad i = 1, 2, \ldots, n. \tag{30}$$

The control $u^*(t)$ is then given by

$$u^*(t) = u_0(t) + \sum_{i=1}^{n} \mu_i(u_i(t) - u_0(t)). \tag{31}$$

## 4. Approximation of the Reachable Set

Consider the system (1) with admissible controls lying in $U_p$ given by (3) and initial state $x(t_0) = 0$. The corresponding reachable set $R(t_1)$ is given in an implicit form by (8). The iterative procedure of the previous section generates a sequence of $n$-simplexes which approximate $R(t_1)$ with increasing accuracy, in the vicinity of a point. The purpose of this section is to develop an iterative procedure which generates two sequences of polyhedra $\{R_k\}$ and $\{N_k\}$ which approximate the entire set $R(t_1)$.

*The Iterative Procedure*

(i) $1 \leqslant k \leqslant n$: Let $R_{k-1} \subset R(t_1)$ be a known $(k-1)$-dimensional polyhedron and let the $(k-1)$-dimensional hyperplane containing $R_{k-1}$ be denoted by $P_{k-1}$. Let $h_k$ be an arbitrary nonzero $n$-vector normal to $P_{k-1}$, i.e.,

$$h_k \perp P_{k-1} \tag{32}$$

and compute the control $u(t, h_k)$ using Eq. (16), (17), or (18). The polyhedron $R_k$ is defined as

$$R_k = \Delta(R_{k-1}, s(h_k), -s(h_k)), \tag{33}$$

where $s(h_k)$ is a contact point of $R(t_1)$ given by

$$s(h_k) = \int_{t_0}^{t_1} \Phi(t_1, t) B(t) u(t, h_k) dt. \tag{34}$$

Define the functions $\Psi_k^1(x)$ and $\Psi_k^2(x)$, and the set $M_k$ as follows:

$$\Psi_k^1(x) = \langle h_k, x\rangle - \eta(h_k), \qquad \Psi_k^2(x) = \langle h_k, x\rangle + \eta(h_k), \tag{35}$$

$$M_k = \{x \in E^n : \Psi_k^i(x) \leqslant 0, i = 1, 2\}. \tag{36}$$

Then the set $N_k$ is defined as

$$N_k = N_{k-1} \cap M_k. \tag{37}$$

(ii) $k > n$: Let $R_{k-1}$ and $N_{k-1}$ be two known $n$-dimensional polyhedra such that $R_{k-1} \subset R(t_1) \subset N_{k-1}$. Assume that $R_{k-1}$ has $I_{k-1}$ faces, $Q_{k-1}^i$, $i = 1, 2,..., I_{k-1}$, which are $(n-1)$-dimensional polyhedra [1]. Let $h_k^i$ be arbitrary nonzero $n$-vectors normal to $Q_{k-1}^i$, i.e.,

$$h_{k-1}^i \perp Q_{k-1}^i, \qquad i = 1, 2,..., I_{k-1}, \tag{38}$$

and compute the controls $u(t, h_k^i)$ using Eq. (16), (17), or (18). Then the polyhedron $R_k$ is defined as

$$R_k = \Delta(R_{k-1}, s(h_k^1), s(h_k^2),..., s(h_k^{I_{k-1}})), \tag{39}$$

where $s(h_k^i)$ are contact points of $R(t_1)$ given by

$$s(h_k^i) = \int_{t_0}^{t_1} \Phi(t_1, t) B(t) u(t, h_k^i) dt, \qquad i = 1, 2,..., I_{k-1}. \tag{40}$$

Let the functions $\Psi_k^i(x)$, $i = 1, 2,..., I_{k-1}$, and the set $M_k$ be defined as follows:

$$\Psi_k^i(x) = \langle h_k^i, x\rangle - \eta(h_k^i), \qquad i = 1, 2,..., I_{k-1}, \tag{41}$$

$$M_k = \{x \in E^n : \Psi_k^i(x) \leqslant 0, i = 1, 2,..., I_{k-1}\}. \tag{42}$$

Then the polyhedron $N_k$ is defined as

$$N_k = N_{k-1} \cap M_k. \tag{43}$$

In order to start the iterative procedure, choose $R_0 = 0$ and $N_0 = E^n$. The following theorem summarizes the essential features of the iterative procedure.

THEOREM 3. *The two sequences of sets* $\{R_k\}$ *and* $\{N_k\}$ *generated by the iterative procedure are such that*

(i) $R_{k-1} \subset R_k \subset R(t_1) \subset N_k \subset N_{k-1}$ *for all* $k \geqslant 1$,

(ii) $\mu(R_k) \to \mu(R(t_1))$, *and* $\mu(N_k) \to \mu(R(t_1))$ *as* $k \to \infty$, *where* $\mu(R)$ *represents the Lebesgue measure of* $R$.

*Remarks.* The inclusion relations in (i) follow directly from the definitions of $R_k$ and $N_k$. $\{R_k\}$ is a monotonically increasing sequence of sets while $\{N_k\}$ is a monotonically decreasing sequence. While both sequences consequently converge, they do not converge to the set $R(t_1)$. The limit of the monotonically decreasing sequence of closed sets is also closed but the limit of the sequence $\{R_k\}$ is neither open nor closed. $\mathrm{Lim}_{k\to\infty} R_k = \bigcup_{i=1}^{\infty} R_i$ and contains a countable number of its limit points belonging to $R(t_1)$. The relation (ii) indicates, however, that if a Lebesgue measure of the sets $R_k$ and $N_k$ is considered, the sequences converge to $\mu(R(t_1))$. If $\mu(R(t_1)) - \mu(R_k) = \Delta\mu(R_k)$, then $\Delta\mu(R_k)$ satisfies the relations $\Delta\mu(R_k) \geqslant 0$ where equality is satisfied only when $\mu(R_k) = \mu(R(t_1))$, and $\Delta\mu(R_k) > \Delta\mu(R_{k+1})$ if $\mu(R_k) \neq \mu(R(t_1))$. Hence, by Lyapunov's theorems, $\Delta\mu(R_k)$ tends to zero and $\lim_{k\to\infty} \mu(R_k) = \mu(R(t_1))$. The same arguments apply to the convergence of $\mu(N_k)$.

*Energy Constraint,* $p = 2$. A common practical constraint imposed on the admissible controls is the limitation of the control energy, which corresponds to $p = 2$ in (3). This is a very special case from the view point of reachability, for in this case it is possible to determine an explicit expression for the reachable set.

Application of the control $u(t, h)$ given by Eq. (18) with $p = 2$ yields the following expression for the boundary point, $s(h)$ of $R(t_1)$

$$s(h) = (c_2/\sqrt{h^T \overline{W}(t_0, t_1) h})\, \overline{W}(t_0, t_1) h, \tag{44}$$

where $\overline{W}(t_0, t_1)$ is the $(n \times n)$ symmetric, nonsingular matrix given by

$$\overline{W}(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, B^T(t)\, \Phi^T(t_1, t)\, dt. \tag{45}$$

Since $R(t_1)$ is convex, every boundary point can be written in the form (44) for some nonzero vector $h$. While this is still an implicit expression for $\partial R(t_1)$ given in terms of the vector $h$, due to the special form of the right side of Eq. (44) the quadratic form $s^T(h)\, \overline{W}^{-1}(t_0, t_1)\, s(h)$ is independent of $h$, viz.,

$$s^T(h)\, \overline{W}^{-1}(t_0, t_1)\, s(h) = c_2^{\,2}. \tag{46}$$

Thus, every point $x \in \partial R(t_1)$ satisfies Eq. (46), which is the equation of an ellipsoid in $E^n$. Consequently, the explicit expression for the reachable set is

$$R(t_1) = \{x \in E^n : x^T \overline{W}^{-1}(t_0, t_1)\, x \leqslant c_2^2\}. \tag{47}$$

## 5. Multiple Control Constraints

In some control problems the admissible controls may have to satisfy more than one constraint of the type (3) simultaneously. In such cases the admissible controls are restricted to lie in the intersection of the individual constraint sets. The developments presented in Sections 3 and 4 are equally applicable, in principle at least, to problems with multiple control constraints. Computationally, however, these problems are more involved. The reason for this is twofold. First, it may be difficult to deduce from the Maximum Principle, or by standard techniques of functional analysis, the structure of the controls $u(t, h)$ which transfer the initial state of a given system to the boundary of $R(t_1)$. Secondly, even if the structure of $u(t, h)$ is clearly evident, the actual controls may depend on unknown parameters whose values can be found, in general, only by an iterative procedure. Therefore, when it is desired to determine whether or not a given state is reachable or to find an approximation of the set $R(t_1)$ for systems whose controls have to satisfy multiple constraints, the algorithms presented in Sections 3 and 4 will involve iterative schemes within iterative schemes. In order to illustrate these ideas the problem of determining the control $u(t, h)$ is solved when the admissible controls must lie in the intersection of $U_2$ and $U_\infty$. This corresponds to the simultaneous limitation of control energy and amplitude.

*Energy and Amplitude Constraints*

Consider system (1) with a set $U$ of admissible controls given by

$$U = U_2 \cap U_\infty = \{u \in L_\infty(t_0, t_1) : \| u \|_2 \leqslant E;\ | u_j(t)| \leqslant M,\ j = 1, 2, \ldots, m \text{ on } (t_0, t_1)\}, \tag{48}$$

where $E$ and $M$ are arbitrary positive numbers. Before the Maximum Principle can be applied to find $u(t, h)$, it must be noted that $u(t, h)$ does not necessarily lie on the boundary of either $U_2$ or $U_\infty$ but is determined by the relative magnitudes of $E$ and $M$. Let $u' \in U_\infty$ be a control which lies on the boundary of $U_\infty$, i.e.,

$$| u_j'(t)| = M, \qquad j = 1, 2, \ldots, m, \quad \text{a.e.} \quad \text{on } (t_0, t_1). \tag{49}$$

The corresponding control "energy" $E'$ is given by

$$E' = \| u' \|_2 = M \cdot \sqrt{m(t_1 - t_0)}. \tag{50}$$

Clearly, if $E > E'$, then the dominating constraint is $U_\infty$, and therefore the control $u(t, h)$ cannot lie on the boundary of $U_2$. On the other hand, if $E < E'$, then $u(t, h)$ can be shown to lie on the boundary of $U_2$. In view of these remarks it can be concluded that the reachable set corresponding to multiple constraints is a subset of the intersection of the reachable sets corresponding to the individual constraints.

Assume that $E < E'$ and let $H(x, \lambda, u)$ denote the Hamiltonian given by

$$H(x, \lambda, u) = \lambda^T(A(t)x + B(t)u(t)) - \mu \sum_{j=1}^{m} | u_j(t)|^2, \tag{51}$$

where $\lambda$ is the adjoint variable and $\mu$ is a Lagrange multiplier introduced to satisfy the constraint $U_2$. It can be easily shown by the use of the Maximum Principle that the control $u(t, h)$ which maximizes $H(x, \lambda, u)$ subject to the constraint (48) is of the form

$$u(t, h) = M \cdot \mathrm{sat}(c \cdot f(t, h)), \tag{52}$$

where $f(t, h)$ is defined by (14), $c = 1/2\mu$, and the sat$(\cdot)$ function is defined as

$$M \cdot \mathrm{sat}(y) = \begin{cases} M & y > M, \\ y & | y | \leqslant M, \\ -M & y < -M. \end{cases} \tag{53}$$

Thus, the problem of determining $u(t, h)$ in Eq. (52) reduces to finding the value of $c = c^*$ such that $E(c^*) = E$, where $E(c)$ is the scalar function

$$E(c) = \left[ \int_{t_0}^{t_1} \sum_{j=1}^{m} | M \cdot \mathrm{sat}(c \cdot f_j(t, h))|^2 \, dt \right]^{1/2}. \tag{54}$$

The gradient method could be used to find $c^*$, but its convergence is generally slow. The faster Newton–Raphson method may not converge at all since $E(c)$ is not a convex function. A definitely convergent process is the binary search method, but it can be used only if two values of $c$ are known which bound $c^*$ from above and below. The procedure that was used in this study is a combination of the Newton–Raphson and binary search methods: Whenever the former is found to diverge, the latter is used to obtain the next value of $c$, and then the Newton–Raphson method is resumed. Computational experience indicates that $c^*$ can be obtained by this procedure in a small number of iterations.

*A Class of Optimal Control Problems*

The iterative procedure presented in Section 3 results in an admissible control function that transfers the initial state of a dynamical system to a point nearest to a prescribed target. If the target is an element of the reachable set $R(t_1)$, then the exact transfer is accomplished. If there exist more than one admissible control which transfer the initial state of the system to the target, then it is reasonable to define a performance index for the system and look for an admissible control which not only transfers the initial state to the target, but also minimizes the performance index. The problem thus becomes an optimal control problem in the usual sense. The purpose of this section is to develop an iterative procedure for the solution of a class of optimal control problems. The procedure is based on the results of Sections 3–5.

*Problem Statement.* Consider the linear dynamical system (1) with initial state $x(t_0) = 0$ and terminal target $x(t_1) = \alpha$, where $\alpha$ is a fixed point in $E^n$. The set of admissible controls $U_p$ is given by (3) for some fixed $p$, $1 \leqslant p \leqslant \infty$, and the performance index $J(u)$ is defined as

$$J[u] = \left[ \int_{t_0}^{t_1} \sum_{j=1}^{m} | u_j(t) |^r \, dt \right]^{1/r} = \| u \|_r \tag{55}$$

for some fixed $r$, $1 \leqslant r \leqslant \infty$. Find the admissible control $u^*(t)$ such that

(i) if $\alpha \in R(t_1)$, then $u^*(t)$ transfers the state of the system to $\alpha$ and at the same time minimizes $J[u]$, or

(ii) if $\alpha \notin R(t_1)$, then $u^*(t)$ transfers the state of the system to a point $x^*$ such that

$$\| \alpha - x^* \| = \min_{x \in R(t_1)} \| \alpha - x \|. \tag{56}$$

The solution of this problem is obtained by an iterative procedure whose first step is the application of the algorithm presented in Section 3 to determine whether or not $\alpha \in R(t_1)$. If at some stage it is found that $\alpha \in R(t_1)$, then the performance index is converted into an additional control constraint, which results in a problem of multiple constraints but free of an explicit performance index.

*The Iterative Procedure*

*Step* 1. Let $R_p(t_1)$ denote the reachable set corresponding to the control set $U_p$; apply the algorithm of Section 3 to determine whether or not $\alpha \in R_p(t_1)$. If $\alpha \notin R_p(t_1)$, then the exact transfer is not possible and the algorithm converges to a point $x^* \in \partial R_p(t_1)$ such that (56) is satisfied. If $\alpha \in \partial R_p(t_1)$, then the transfer is possible and the algorithm converges to the point $x^* = \alpha \in \partial R_p(t_1)$. In either case, the solution $u^*(t)$ is that control which transfers the state of the system to $x^*$, and $J[u^*]$ is the corresponding value of the performance index.

If $\alpha \in \operatorname{int} R_p(t_1)$, then at some stage, say $j = \bar{v}$, of the algorithm, $\alpha \in \operatorname{int} S_{\bar{v}}$, where $S_{\bar{v}}$ is the closed convex hull of $(n + 1)$ points $y_0, y_1, \dots, y_n \in R_p(t_1)$ corresponding to the admissible controls $u_0, u_1, \dots, u_n \in U_p$. Consequently, $\alpha$ can be expressed as

$$\alpha = y_0 + \sum_{i=1}^{n} \mu_i(y_i - y_0), \tag{57}$$

where the $\mu_i$ are real numbers satisfying the inequalities

$$\sum_{i=1}^{n} \mu_i < 1, \qquad \text{and} \qquad 0 < \mu_i < 1, \qquad i = 1, 2, \dots, n. \tag{58}$$

Let $u^1(t)$ be the control defined by

$$u^1(t) = u_0(t) + \sum_{i=1}^{n} \mu_i(u_i(t) - u_0(t)). \tag{59}$$

*Step* 2. At the $k$-th stage of the iterative procedure, let $u^k(t)$ be an admissible control that transfers the initial state of the system to $\alpha$, i.e.,

$$\alpha = \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, u^k(t)\, dt, \qquad u^k \in U_p. \tag{60}$$

Compute $J[u^k]$ and denote its value by $c^k$, i.e.,

$$J[u^k] = c^k. \tag{61}$$

Define the constraint set $U_r^k$ and the corresponding reachable set $R_r^k(t_1)$ as follows:

$$U_r^k = \{u \in L_r(t_0, t_1) : \| u \|_r \leqslant c^k\}, \tag{62}$$

$$R_r^k(t_1) = \left\{x \in E^n : x = \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, u(t)\, dt,\ u \in U_r^k\right\}. \tag{63}$$

Let $U^k$ denote the intersection of $U_p$ and $U_r^k$, and let $R^k(t_1)$ be the corresponding reachable set, i.e.,

$$U^k = U_p \cap U_r^k, \tag{64}$$

$$R^k(t_1) = \left\{x \in E^n : x = \int_{t_0}^{t_1} \Phi(t_1, t)\, B(t)\, u(t)\, dt,\ u \in U^k\right\}. \tag{65}$$

Clearly, $\alpha \in R^k(t_1)$.

*Step* 3. Apply the algorithm of Section 3 to $R^k(t_1)$. If $\alpha \in \partial R^k(t_1)$, then the algorithm converges to the point $x^* = \alpha \in \partial R^k(t_1)$, and the solution of the problem is $u^*(t) = u^k(t)$, with $J[u^*] = c^* = c^k$.

If $\alpha \in \operatorname{int} R^k(t_1)$, then at some stage, say $j = v_k$, of the algorithm, $\alpha \in \operatorname{int} S_{v_k}$, where $S_{v_k}$ is the closed convex hull of $(n + 1)$ points $y_0, y_1, \ldots, y_n \in R^k(t_1)$ corresponding to the admissible controls $u_0, u_1, \ldots, u_n \in U^k$. Consequently, $\alpha$ can be expressed in the form (57) for some real numbers $\mu_i$, $i = 1, 2, \ldots, n$, satisfying the inequalities (58). Define the control $u^{k+1}(t)$ as

$$u^{k+1}(t) = u_0(t) + \sum_{i=1}^{n} \mu_i(u_i(t) - u_0(t)) \tag{66}$$

and return to Step 2.

The following theorem summarizes the results of the iterative procedure.

THEOREM 4. *The sequence of control functions* $\{u^k(t)\}$ *generated by the iterative procedure is such that*

(i) $u^k(t)$, $k \geqslant 1$, *is admissible and accomplishes the desired transfer,*

(ii) $J[u^{k+1}] \leqslant J[u^k]$, $k \geqslant 1$, *where the equality sign holds if and only if* $u^k(t) = u^{k+1}(t) = u^*(t)$,

(iii) $u^k(t) \to u^*(t)$.

## 6. FEEDBACK PARAMETERS

The results and iterative procedures developed in the previous sections were related to the reachable sets for dynamical systems of fixed structure. The purpose of the present section is to investigate the reachable sets for systems which contain some adjustable parameters. The systems under consideration are linear, autonomous, and are assumed to be completely controllable, and the admissible controls are bounded and measurable functions. The adjustable parameters arise through the introduction of a linear feedback of the state.

Consider the completely controllable system described by the equation

$$\dot{x} = (A - bk^T)\, x + bu = A(k)\, x + bu, \tag{67}$$

where $A$ and $b$ are in the companion form

$$A = \left[\begin{array}{c|cccc} & 1 & & & \\ & & 1 & 0 & \\ 0 & & & \ddots & \\ & & 0 & & 1 \\ \hline -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{array}\right] \qquad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{68}$$

$k$ is an $n$-vector of feedback gains whose components can be adjusted, and $u$ is a scalar control said to be admissible if it is an element of the set

$$U \subset L_\infty(0, t_1)$$

given by

$$U = \{u \in L_\infty(0, t_1) : |\, u(t)| \leqslant 1 \text{ for all } t \in (0, t_1)\}. \tag{69}$$

Let $R(t_1\,; k)$ denote the reachable set for system (67) with control set (69). It is well known that the eigenvalues of system (67) can be adjusted arbitrarily by a suitable choice of the vector $k$. This, together with the results of Theorem 2, implies that the set $R(t_1\,; k)$ can be bounded for all $t_1 \geqslant 0$ by a proper choice of the feedback gain vector $k$. Once a vector $k$ is chosen such that the eigenvalues of $A(k)$ have negative real parts, the bound on $R(t_1\,; k)$ in any particular direction can be obtained by the straightforward application of the results of Section 3. The inverse problem, however, of determining a feedback gain vector $k$ given a desired bound on $R(t_1\,; k)$ in any direction is nontrivial and is encountered in the following problem.

*Problem.* Let $h$ be an arbitrary nonzero $n$-vector and $d$ a positive constant. Given the feedback system (67) with control set $U$ defined by (69) and initial state $x(0) = 0$, find an $n$-vector $k^*$ of feedback gains such that the state of the system remains in the closed half space defined by

$$T = \{x \in E^n : \langle h, x\rangle \leqslant d\} \tag{70}$$

for all $t > 0$ regardless of the control applied.

In view of the results of Section 3 it is clear that $k^*$ is a solution of the problem if and only if

$$\eta(h; k^*) \leqslant d, \tag{71}$$

where $\eta(h; k)$ is the support function of $R(\infty; k)$ given by

$$\eta(h; k) = \langle h; s(h; k)\rangle = \int_0^\infty f(t, h; k) \operatorname{sgn}(f(t, h; k))\, dt = \int_0^\infty |f(t, h; k)|\, dt, \tag{72}$$

and $f(t, h; k)$ is the scalar function defined as

$$f(t, h; k) = h^T e^{A(k)t} b. \tag{73}$$

The determination of a vector $k^*$ such that (71) is satisfied is, in general, a very difficult problem. In the sequel, a solution $k^*$ will be obtained for the case when the vector $h$ coincides with any one of the coordinate axes of the state space.

Select a set of $n$ real, distinct, negative numbers $\bar{\lambda}_j$, $j = 1, 2,..., n$, and compute the feedback gain vector $\bar{k}$ such that the eigenvalues of the matrix $A(\bar{k})$ are precisely these numbers. Since $A$ and $b$ are in companion form, it can be easily shown that the expression for $\eta(h; \bar{k})$ is

$$\eta(h; \bar{k}) = \int_0^\infty \left| \sum_{j=1}^{n} \bar{a}_j e^{\bar{\lambda}_j t} \right| dt, \tag{74}$$

where the coefficients $\bar{a}_j$ are given by the residues, viz.,

$$\bar{a}_j = \frac{h_1 + h_2\bar{\lambda}_j + h_3\bar{\lambda}_j^2 + \cdots + h_n\bar{\lambda}_j^{n-1}}{(\bar{\lambda}_j - \bar{\lambda}_1) \cdots (\bar{\lambda}_j - \bar{\lambda}_{j-1})(\bar{\lambda}_j - \bar{\lambda}_{j+1}) \cdots (\bar{\lambda}_j - \bar{\lambda}_n)}, \quad j = 1, 2,..., n. \tag{75}$$

If $\eta(h; \bar{k}) \leqslant d$, then $\bar{k}$ is a solution; if $\eta(h; \bar{k}) > d$, then a new feedback gain vector $\tilde{k}$ must be selected such that

$$\eta(h; \tilde{k}) = c \cdot \eta(h; \bar{k}), \tag{76}$$

where $c$ is a positive constant given by $c = d/\eta(h; \bar{k})$. Assume that $\tilde{k}$ is chosen such that the resulting eigenvalues $\tilde{\lambda}_j$, $j = 1, 2,..., n$ of $A(\tilde{k})$ are again real, distinct, and negative. Consequently, $\eta(h; \tilde{k})$ takes the same form as $\eta(h; \bar{k})$ except that $\bar{\lambda}_j$ is replaced by $\tilde{\lambda}_j$, and $\bar{a}_j$ by $\tilde{a}_j$. A set of sufficient conditions for (76) to be satisfied is

$$\begin{array}{ll} \text{(i)} & \tilde{\lambda}_j = c_1\bar{\lambda}_j \\ \text{(ii)} & \tilde{a}_j = c_2\bar{a}_j \end{array} \Bigg\} \quad j = 1, 2,..., n,$$

where $c_1$ and $c_2$ are constants. Equation (76) holds if and only if $c_1$ and $c_2$ satisfy the relation

$$c_2/c_1 = c. \tag{77}$$

Combining condition (i) with Eqs. (75) and (77), condition (ii) can be written in the following matrix form

$$\begin{bmatrix} 1 & \bar{\lambda}_1 & \bar{\lambda}_1^2 & \cdots & \bar{\lambda}_1^{n-1} \\ 1 & \bar{\lambda}_2 & \bar{\lambda}_2^2 & \cdots & \bar{\lambda}_2^{n-1} \\ \vdots & & & & \\ 1 & \bar{\lambda}_n & \bar{\lambda}_n^2 & \cdots & \bar{\lambda}_n^{n-1} \end{bmatrix} \begin{bmatrix} (1/c_1^{\,n} - c)h_1 \\ (1/c_1^{n-1} - c)h_2 \\ \vdots \\ (1/c_1 - c)h_n \end{bmatrix} = 0. \tag{78}$$

Since the $\bar{\lambda}_j$ , $j = 1, 2,..., n$, are distinct, the Vandermonde matrix in (78) is nonsingular. This implies that

$$(1/c_1^{n+1-j} - c)h_j = 0 \qquad \text{for all} \quad j = 1, 2,..., n. \tag{79}$$

Since $h$ is a nonzero vector, (79) can be satisfied for all $j$ only if $h$ has one nonzero component, say $h_i \neq 0$, in which case $c_1$ is given by

$$c_1 = (1/c)^{1/n+1-i} = (\eta(h; \bar{k})/d)^{1/n+1-i}. \tag{80}$$

Thus, if $h$ coincides with the $i$-th coordinate axis of the state space, then $k^* = \bar{k}$ is a solution to the problem, and the $i$-th component of the state vector remains bounded for all time regardless of what admissible control is applied. By computing the support functions $\eta(h; \bar{k})$ for $n$ different vectors $h$ that coincide with the $n$ coordinate axes, the above method enables one to place an upper bound on the magnitude of each component of the state *simultaneously*. Thus, the infinite time-reachable set can be made to lie within a specified parallelepiped by a suitable choice of the feedback gain vector.

The restriction to distinct eigenvalues is not necessary. On the other hand, the relaxation to complex eigenvalues, as well as choices of the vector $h$ not along the coordinate axes, would make the above analysis quite intractable.

## 7. Illustrative Examples

In this section computational results are presented to illustrate the applications of the various methods developed in Sections 3–6.

*Example* 1

A dynamical system with two control inputs is described by the fourth-order vector differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0.5 & 1.2 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

TABLE I

Numerical Results of the Algorithm for System of Example 1

| $\alpha, x^*$ | $k$ | $\Vert h_k \Vert$ | $\langle h_k, \alpha - s(h_k)\rangle$ | $\min_k$ | $\Vert h_k \Vert - \min_k$ |
|---|---|---|---|---|---|
| | 1 | 2.000000 | 1.348013 | 0.674007 | 1.325993 |
| $\alpha = \begin{bmatrix} 1.000000 \\ 1.000000 \\ -1.000000 \\ 1.000000 \end{bmatrix}$ | 2 | 1.130712 | −0.671264 | 0.000000 | 1.130712 |
| | 3 | 1.048930 | 0.897893 | 0.856048 | 0.192882 |
| | 4 | 1.018998 | 0.995522 | 0.976962 | 0.042036 |
| | 5 | 0.999153 | 0.987073 | 0.987909 | 0.011244 |
| | 6 | 0.997706 | 0.984590 | 0.986853 | 0.010853 |
| | 7 | 0.989848 | 0.976805 | 0.986823 | 0.003026 |
| $x^* = \begin{bmatrix} 0.794797 \\ 0.458755 \\ -0.203143 \\ 1.092619 \end{bmatrix}$ | 8 | 0.989648 | 0.978642 | 0.988879 | 0.000770 |
| | 9 | 0.989400 | 0.978694 | 0.989180 | 0.000221 |
| | 10 | 0.989281 | 0.978536 | 0.989138 | 0.000143 |
| | 11 | 0.989274 | 0.978619 | 0.989229 | 0.000045 |
| | 12 | 0.989263 | 0.978590 | 0.989211 | 0.000052 |
| | 13 | 0.989249 | 0.978607 | 0.989242 | 0.000008 |
| $\alpha = \begin{bmatrix} 0.300000 \\ 0.600000 \\ 0.320000 \\ 0.450000 \end{bmatrix}$ | 1 | 0.868850 | −1.818925 | 0.000000 | 0.868850 |
| | 2 | 0.461043 | −0.463191 | 0.000000 | 0.461043 |
| | 3 | 0.022966 | −0.007715 | 0.000000 | 0.022966 |
| | 4 | 0.015764 | −0.003422 | 0.000000 | 0.015764 |
| $x^* = \alpha$ | 5 | 0.000000 | | | |
| | 1 | 1.460648 | −0.016600 | 0.000000 | 1.460648 |
| $\alpha = \begin{bmatrix} 0.780509 \\ 0.084998 \\ -0.455028 \\ 1.144563 \end{bmatrix}$ | 2 | 0.295352 | −0.908648 | 0.000000 | 0.295352 |
| | 3 | 0.110324 | −0.029491 | 0.000000 | 0.110324 |
| | 4 | 0.053541 | −0.002476 | 0.000000 | 0.053541 |
| | 5 | 0.021704 | −0.000072 | 0.000000 | 0.021704 |
| | 6 | 0.005947 | −0.000002 | 0.000000 | 0.005947 |
| $x^* = \alpha$ | 7 | 0.001189 | −0.000000 | 0.000000 | 0.001189 |
| | 8 | 0.000397 | 0.000000 | 0.000000 | 0.000396 |
| | 9 | 0.000000 | | | |

The initial state of the system is $x(0) = 0$ and the admissible controls are assumed to be limited in amplitude: $| u_1 | \leqslant 1 \; | u_2 | \leqslant 1$. The initial time $t_0 = 0$ and the final time $t_1 = 1$. The algorithm developed in Section 3 was applied to this system to determine the points $x^*$ and control functions $u^*(t)$ for various terminal states $\alpha$. The results are shown in Table I and the

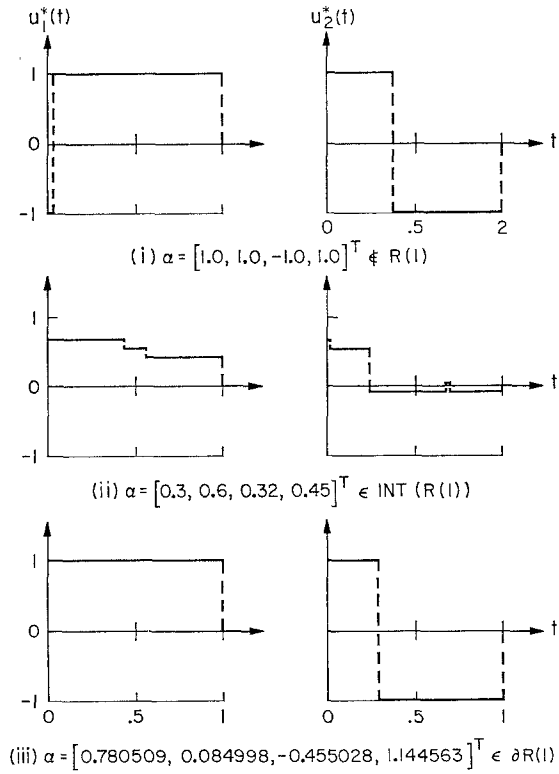


FIG. 1. The control functions $u^*(t)$ for Example 1.

corresponding control functions are shown in Fig. 1. The quantity $\min_k$ in Table I is defined as

$$\min_k = \max \left[ 0, \frac{\langle h_k, \alpha - s(h_k) \rangle}{\| h_k \|} \right]$$

and represents a lower bound for the quantity $\| \alpha - x^* \|$.

*Example* 2 (*Approximations of Reachable Sets*)

The iterative procedure of Section 4 is applied to the following timevarying system to obtain approximations of the reachable sets for several values of the terminal time $t_1$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (e^{-t} \sin t - 2) & te^{-2t} \\ -e^{-t} & (2e^{-2t} \cos t - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \sin t \end{bmatrix} u$$

$$| u(t) | \leqslant 1 \quad \text{in} \quad [0, t_1].$$

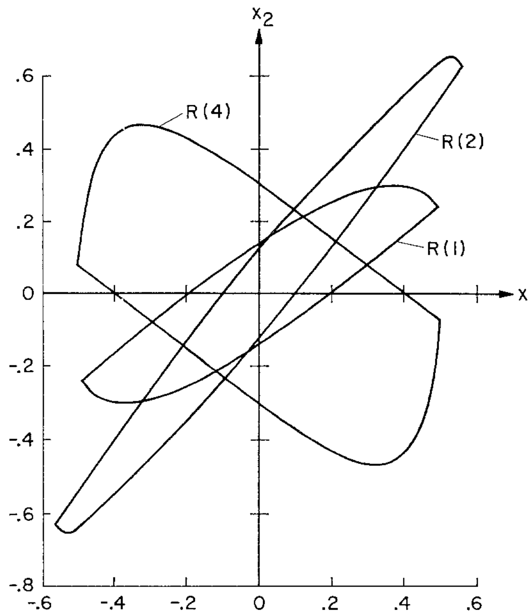


FIG. 2. Approximations of $R(t_1)$ for a nonautonomous system (Example 2).

TABLE II (Example 2)

| $k$ | $V_2(R_k)$ | $V_2(N_k)$ | $V_2(R_k)/V_2(N_k)$ | $V_2(N_k)/V_2(R_k)$ |
|---|---|---|---|---|
| 2 | 0.414393 | 0.828786 | 0.500000 | 2.000000 |
| 3 | 0.487036 | 0.538810 | 0.903911 | 1.106304 |
| 4 | 0.502928 | 0.512560 | 0.981207 | 1.019153 |
| 5 | 0.507739 | 0.509983 | 0.995599 | 1.004421 |

Figure 2 shows the approximations of $R(t_1)$ for $t_1 = 1, 2$, and 4. It is clear from the figure that the growth of the reachable set $R(t_1)$ need not be monotonic with respect to time for nonautonomous systems. If $V_2(R_k)$ and $V_2(N_k)$ are the areas of the sets $R_k$ and $N_k$, the accuracy of the approximations may be computed by the ratios of these quantities which must tend to unity as $k$ increases. These ratios for $t_1 = 4$ are indicated in Table II.

*Example* 3 (*Joint amplitude and energy constraints*)

Using the iterative procedure of Section 5, the following problem having two constraints is considered:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$| u(t) | \leqslant 1, \qquad \left[ \int_0^4 u^2(t)\, dt \right]^{1/2} \leqslant E, \qquad 0 \leqslant t \leqslant 4.$$

Under the dual constraints it is desired to determine the reachable set $R(4)$. Figure 3 shows the approximations after five iterations of $R(4)$ with joint

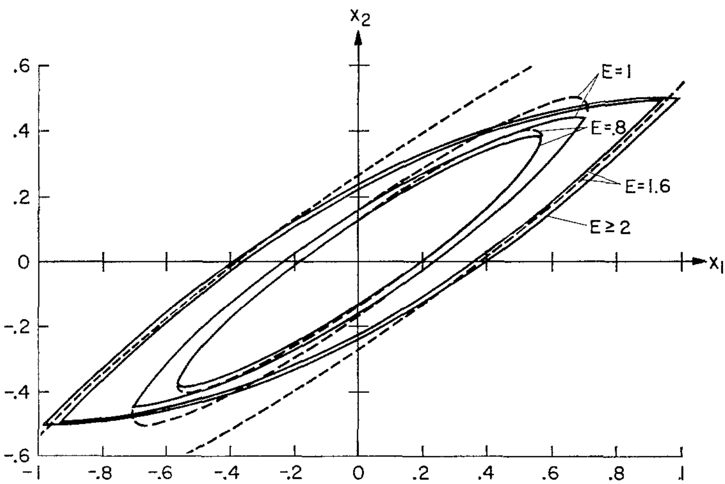


FIG. 3. Approximations of $R(4)$ with joint amplitude and energy constraints (Example 3).

amplitude and energy constraints for values of $E = 0.8$, 1.0, and 1.6. For comparison purposes the reachable sets corresponding to the individual constraints are also included. The outermost solid curve is the boundary of the reachable set with only the amplitude constraint while the dashed curves represent reachable sets with only energy constraints. For $E \geqslant 2$ only, the amplitude constraint is effective. Since the sets are approximated after 5 iterations they are closed convex hulls of $2^5 = 32$ boundary points.

*Example* 4 (*System with adjustable parameters*)

In a system described by the equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$k_1$ and $k_2$ are adjustable parameters. The control function $u$ is subject to the amplitude constraint $| u(t)| \leqslant 1$. It is desired to find the values of the parameters $\bar{k} = [k_1, k_2]$ such that for any admissible control function, the state of the system remains within a bounded region $T$ defined by

$$T = \{x \in E^n : | x_1 | \leqslant d_1, | x_2 | \leqslant d_2\}$$

for all time $t > 0$.

A set of eigenvalues $\bar{\gamma} = [\gamma_1, \gamma_2]$ is first chosen for the system. $\gamma_1 = -0.5$, $\gamma_2 = -1.0$, so that the system is asymptotically stable. The corresponding values of the parameters are found to be $\bar{k} = [1.5, 1.5]$ and the reachable set $R(\infty, \bar{k})$ is shown in Fig. 4. The corresponding support functions are found

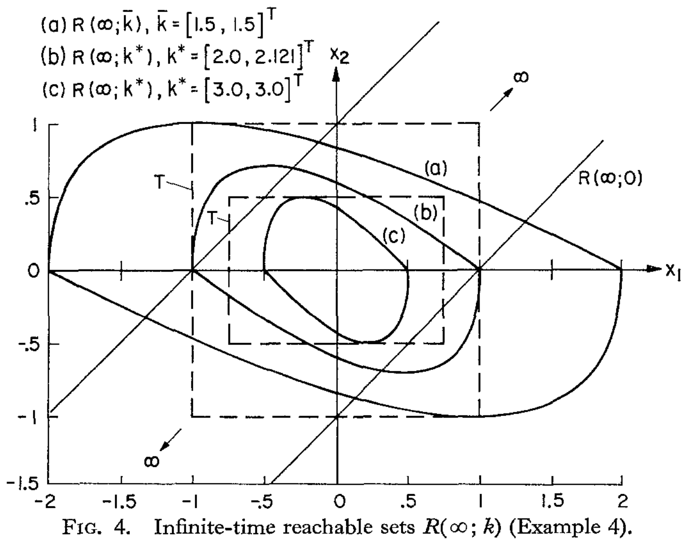


FIG. 4. Infinite-time reachable sets $R(\infty; k)$ (Example 4).

to be $\eta(h^1; \bar{k}) = 2.0$, $\eta(h^2; \bar{k}) = 1.0$. We consider the following two cases for the constraint on the reachable set:

(i) $d_1 = d_2 = 1.0$; Eq. (80) yields $c_1 = \sqrt{2}$ and $c_2 = 1$, and hence $c_{\max} = \sqrt{2}$. Choosing the new eigenvalues to be $\gamma^* = \sqrt{2}\,\bar{\gamma} = [-.707, -1.414]$, the parameter values are found to be $k^* = [2.0, 2.121]$.

(ii) $d_1 = 0.75$, $d_2 = 0.5$; Eq. (80) yields in this case $c_1 = 1.632$, $c_2 = 2$. Selecting $\gamma^* = 2\bar{\gamma} = [-1.0, -2.0]$, the parameter values are found to be $k^* = [3.0, 3.0]$.

Figure 4 shows the sets $R(\infty; 0)$, $R(\infty; \bar{k})$ as well as the sets $R(\infty; k^*)$ for the two cases (i) and (ii). Since the initial choice of the eigenvalues is arbitrary (except for negative real parts), this procedure yields only sufficient conditions to satisfy the constraints.

## References


Alexandroff, P. (1961), "Elementary Concepts of Topology," Dover, New York.

Barr, R. O. (1967), Computation of optimal controls on convex reachable sets, *in* "Mathematical Theory of Control," Academic Press, New York.

Barr, R. O. (1969), An efficient computational procedure for a generalized quadratic programming problem, *SIAM J. Control* **7**, 415–429.

Formalskii, A. M. (1967), Construction of the region of controllability for systems with bounded-impulse controls, *SIAM J. Control* **5**, 211–221.

Gilbert, E. G. (1966), An iterative procedure for computing the minimum of a quadratic form on a convex set, *SIAM J. Control Ser. A*, **4**, 61–80.

MeMay, J. L. (1964), Recoverable and reachable zones for control systems with linear plants and bounded controller outputs, Proceedings of the Joint Automatic Control Conference, Stanford University, Stanford, Calif.

Neustadt, L. W. (1963). The existence of optimal controls in absence of convexity conditions, *J. Math. Anal. Appl.* **7**, 110–117.

Pecsvaradi, T. and K. S. Narendra (1970), A new iterative procedure for the minimization of a quadratic form on a convex set, *SIAM J. Control* **8**, 396–402.

Pecsvaradi, T. (1970), "Reachable Sets for Linear Dynamical Systems," Doctoral dissertation, Yale University, New Haven, Conn.

Roxin, E. O. (1960), Reachable zones in autonomous differential systems, *Bol. Soc. Mat. Mexicana* **5**, 125–135.